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Anomalous diffusion in linear shear flows

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Abstract. Anomalous diffusion in the presence of several linear flows is studied by means of a nonlinear diffusion formalism. The results generated are particularly interesting for simple shear flows. We compare our results for this latter situation with those obtained from an alternative description for anomalous diffusion, namely fractional diffusion.

Nonlinear diffusion has been attracting the attention of many researchers in connection with diffusion on fractals [1], diffusion in plasmas [2] and transport in porous media [3]. The kind of nonlinear diffusion which will be studied here has the remarkable property of being directly derived from well known equations of macroscopic physics such as the continuity equation, Darcy's law for flows in porous media and the equation of state of the polytrope [4]. This allows for an immediate identification of physical instances where it might accurately describe the diffusion process. Furthermore, this nonlinear diffusion has been recently related to a new statistical mechanics formalism which claims to describe the statistics of random media [5, 6]. These works suggest that nonlinear diffusion might play a relevant role as a description of anomalous diffusion (i.e. diffusion in which the square of displacement is not proportional to time t but to some real power of t). Other alternative formalisms, such as fractional dynamics [7–9], show a close connection with anomalous diffusion as well. In this case, though, the foundations are not to be found in macroscopic physics but in the stochastic details of the process giving rise to diffusion. In [8] it has been proved that such a diffusion equation is the natural dynamic equation of Lévy flights, the most elegant and natural generalization of Brownian motion. It may prove useful to explore which of these two formalisms provides a better description of anomalous diffusion under some prescribed conditions.

Here we shall study the diffusion of added substances in a shear flow. This is a topic of enormous environmental and industrial importance [10] and is of considerable interest in polymer physics and colloid science as well. Indeed, the presence of a simple shear changes the behaviour $\langle x^2\rangle \sim t$ to $\langle x^2\rangle \sim t^3$ (with x the displacement in the direction of the flow). Much attention has already been paid to this subject by including anisotropies, gravitational effects and sources or sinks [10] but never has anomalous diffusion been allowed into the scheme. In this paper we deal with nonlinear diffusion in linear shear flows. We also

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consider dimensional arguments to draw conclusions on the applicability of the fractional derivatives formalism to the analysis of diffusion in shear flows.

The nonlinear diffusion we are interested in is, in a static fluid,

$$\frac{\partial n}{\partial t} = D\nabla^2 n^q \tag{1}$$

where $n(\boldsymbol{x}, t)$ is the density of diffusing substance, D the diffusion constant and q a parameter whose departure from unity indicates the degree of anomality in diffusion. Equation (1) has been solved in [1, 6] for an initial delta distribution and it has been found that it yields a characteristic time scaling of dispersion of the form

$$\langle \boldsymbol{x}^2 \rangle \sim t^{\frac{2}{3q-1}} \tag{2}$$

in a three-dimensional space and

$$\langle \boldsymbol{x}^2 \rangle \sim t^{\frac{1}{q}} \tag{3}$$

in a two-dimensional space. Both (2) and (3) clearly show the anomalous diffusive properties of equation (1).

We shall now study the following nonlinear diffusion-advection equation

$$\frac{\partial n}{\partial t} + \nabla(n\boldsymbol{v}) = D\nabla^2 n^q \tag{4}$$

for the case of a linear flow $\boldsymbol{v} = \boldsymbol{A}(t) \cdot \boldsymbol{x}$ in an infinite medium.

In the same spirit of [10, 11] we propose a solution in the form

$$n(\boldsymbol{x}, t) = B(t)\left[1 + \frac{1-q}{2}(\boldsymbol{x} - \bar{\boldsymbol{x}}) \cdot \boldsymbol{\sigma}^{-1} \cdot (\boldsymbol{x} - \bar{\boldsymbol{x}})\right]^{\frac{1}{q-1}} \qquad q < 1 \tag{5}$$

where the quantities $B(t)$, $\bar{\boldsymbol{x}}(t)$ and $\boldsymbol{\sigma}(t)$ (the latter is taken to be a symmetric matrix without loss of generality) must be determined in terms of the shear rate $\boldsymbol{A}$ and the diffusion constants D and q.

Substitution of (5) into (4) leads to the following three differential equations:

$$\frac{\mathrm{d}B}{\mathrm{d}t} = -B\,\mathrm{tr}(\boldsymbol{A}) - qDB^q\,\mathrm{tr}(\boldsymbol{\sigma}^{-1}) \tag{6}$$

$$\frac{\mathrm{d}\bar{\boldsymbol{x}}}{\mathrm{d}t} = \boldsymbol{A} \cdot \bar{\boldsymbol{x}} \tag{7}$$

$$\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}t} = \boldsymbol{A} \cdot \boldsymbol{\sigma} + [\boldsymbol{A} \cdot \boldsymbol{\sigma}]^T + 2qDB^{q-1}\mathbf{1}. \tag{8}$$

Furthermore, we shall demand of $n(\boldsymbol{x}, t)$ to be normalized to unity throughout the time evolution:

$$\int n(\boldsymbol{x}, t)\,\mathrm{d}^N\boldsymbol{x} = \int B(t)\left[1 + \frac{1-q}{2}(\boldsymbol{x} - \bar{\boldsymbol{x}}) \cdot \boldsymbol{\sigma}^{-1}(\boldsymbol{x} - \bar{\boldsymbol{x}})\right]^{\frac{1}{q-1}} \qquad \mathrm{d}^N\boldsymbol{x} = 1 \tag{9}$$

where N is the dimension of the space where diffusion takes place.

The integral in (9) is computed by changing to a new set of space coordinates $\boldsymbol{y}$ such that $(\boldsymbol{x} - \bar{\boldsymbol{x}}) \cdot \boldsymbol{\sigma} \cdot (\boldsymbol{x} - \bar{\boldsymbol{x}}) = \boldsymbol{y} \cdot \boldsymbol{y}$ and then using spherical coordinates. This recasts (9) into

$$\Omega_N \sqrt{\det \boldsymbol{\sigma}}\, B \left(\frac{1-q}{2}\right)^{-N/2} \int_0^\infty r^{N-1}(1 + r^2)^{\frac{1}{q-1}} \qquad \mathrm{d}r = 1 \tag{10}$$

where $\Omega_N = 2\pi^{N/2}/\,\Gamma(N/2)$ is the surface of a $(N-1)$-dimensional sphere of unit radius and $\Gamma(x)$ stands for the Gamma function of real argument x.

Denoting with $I_N(q)$ the integral in (10), one finds

$$B(t) = \frac{1}{\Omega_N I_N(q)} \left(\frac{1-q}{2}\right)^{N/2} \sqrt{\det \boldsymbol{\sigma}^{-1}}. \tag{11}$$

In particular, for $N = 2$ and $N = 3$ one obtains

$$B(t) = \frac{q}{2\pi} \sqrt{\det \boldsymbol{\sigma}^{-1}} \tag{12}$$

and

$$B(t) = \left(\frac{q}{2\pi}\right)^{3/2} \frac{\Gamma\left(\frac{q}{1-q}\right)}{\sqrt{\frac{q}{1-q}}\,\Gamma\left(\frac{q}{1-q} - \frac{1}{2}\right)} \sqrt{\det \boldsymbol{\sigma}^{-1}} \qquad q > \tfrac{1}{3} \tag{13}$$

respectively.

Now, with the definition (11) for $B(t)$, it is easy to prove that equation (6) is satisfied and we are actually left with two uncoupled differential equations, namely (7) and (8), which are to be solved for a given $\boldsymbol{A}(t)$.

Before considering a particular case, we make one further general remark regarding the physical meaning of the vector $\bar{\boldsymbol{x}}(t)$ and the matrix $\boldsymbol{\sigma}(t)$: on one hand, $\bar{\boldsymbol{x}}(t)$ is easily seen to correspond to the mean position at time t by explicitly performing the integration of $\boldsymbol{x}n(\boldsymbol{x}, t)$ over space with the aid of the change of variable $\mathbf{y} = \boldsymbol{x} - \bar{\boldsymbol{x}}$. On the other hand, a direct computation of $\boldsymbol{\sigma}^{-1} \cdot \langle(\boldsymbol{x} - \bar{\boldsymbol{x}})(\boldsymbol{x} - \bar{\boldsymbol{x}})\rangle$ leads to the result

$$\boldsymbol{\sigma}^{-1} \cdot \langle(\boldsymbol{x} - \bar{\boldsymbol{x}})(\boldsymbol{x} - \bar{\boldsymbol{x}})\rangle = \frac{1}{1 + \frac{N+2}{2}(q-1)} \mathbf{1} \tag{14}$$

whence one concludes that $\boldsymbol{\sigma}$ and the matrix of correlations $\langle \boldsymbol{x}\boldsymbol{x}\rangle$ are linearly related

$$\langle \boldsymbol{x}\boldsymbol{x}\rangle = \frac{1}{1 + \frac{N+2}{2}(q-1)} \boldsymbol{\sigma} + \bar{\boldsymbol{x}}\bar{\boldsymbol{x}}. \tag{15}$$

Thus, by solving equation (8) for $\boldsymbol{\sigma}$ we directly obtain information about the second moments of the distribution and, in particular, we obtain $\langle \boldsymbol{x}^2\rangle$ and the exponent of anomalous diffusion.

We now turn to an incompressible stationary linear shear flow in two dimensions, such that

$$\boldsymbol{A} = \begin{pmatrix} 0 & G \\ \epsilon G & 0 \end{pmatrix} \tag{16}$$

where G is constant and measures the shear rate and ϵ is a parameter ranging from -1 to 1 and giving rise to different kinds of shear flows, as pure rotation ($\epsilon = -1$), simple shear ($\epsilon = 0$), and pure elongation ($\epsilon = 1$).

The initial condition we will impose to study the diffusion of a point-like drop of substance is $n(\boldsymbol{x}, t = 0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$. A δ-function initial condition is equivalent to $\boldsymbol{\sigma}(t = 0) = 0$ as discussed in [11]. Furthermore, since the initial concentration is centred around $\boldsymbol{x} = \boldsymbol{x}_0$, we have $\bar{\boldsymbol{x}}(t = 0) = \boldsymbol{x}_0$ whence, from (7)

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{\epsilon}Gt) & \frac{1}{\sqrt{\epsilon}} \sinh(\sqrt{\epsilon}Gt) \\ \sqrt{\epsilon} \sinh(\sqrt{\epsilon}Gt) & \cosh(\sqrt{\epsilon}Gt) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{17}$$

for $-1 < \epsilon < 1$. The mean position is therefore variable with time unless one sets $\boldsymbol{x}_0 = 0$. We now study it for different cases.

For a negative ϵ ($-1 < \epsilon < 0$) the mean position $\bar{\boldsymbol{x}}(t)$ follows an ellipse centred at the origin of the x–y plane

$$x(t)^2 + \frac{1}{|\epsilon|} y(t)^2 = x_0^2 + \frac{1}{|\epsilon|} y_0 \tag{18}$$

as might be seen from (17) with $-1 < \epsilon < 0$ and, therefore, the mean position of the tracer particles moves in a periodic trajectory around the origin. For the case $\epsilon = -1$ this ellipse turns into a circle.

For positive ϵ the centre-of-mass of the tracer distribution $\bar{\boldsymbol{x}}(t)$ moves rapidly away from the origin along the asymptotic direction $(1, \sqrt{\epsilon})$ as is found by diagonalizing (17) and taking the eigenvector with nonvanishing asymptotic eigenvalue.

When we set $\epsilon = 0$, $\bar{\boldsymbol{x}}(t)$ moves uniformly along the direction $y(t) = y_0$.

Now, turning to $\boldsymbol{\sigma}$ under these circumstances, the equations for the components of $\boldsymbol{\sigma}$ are, according to (8) and (12),

$$\frac{\mathrm{d}\sigma_{xx}}{\mathrm{d}t} = 2G\sigma_{xy} + 4\pi D \left(\frac{q}{2\pi}\right)^q (\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2)^{\frac{1-q}{2}} \tag{19}$$

$$\frac{\mathrm{d}\sigma_{xy}}{\mathrm{d}t} = G\sigma_{yy} + \epsilon G\sigma_{xx} \tag{20}$$

$$\frac{\mathrm{d}\sigma_{yy}}{\mathrm{d}t} = 2\epsilon G\sigma_{xy} + 4\pi D \left(\frac{q}{2\pi}\right)^q (\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2)^{\frac{1-q}{2}} \tag{21}$$

which are to be solved with the initial conditions $\sigma_{xx}(t = 0) = \sigma_{xy}(t = 0) = \sigma_{yy}(t = 0) = 0$. This system of coupled differential equations does not admit an analytic general solution and we will study it for different types of linear shear flows by setting $\epsilon = -1, 0, 1$.

1. Pure rotational flow ($\epsilon = -1$). In a pure rotational flow, no direction is singled out by any means and we must therefore assume that $\sigma_{xx}(t) = \sigma_{yy}(t)$. From (20) we then conclude that $\sigma_{xy}(t) = 0$ and equations (19) and (21) turn out to be the same, confirming our hypothesis that $\sigma_{xx}(t) = \sigma_{yy}(t)$. The set of equations we are now confronted with is exactly the same, with the same set of initial conditions, as if we had set $G = 0$ from the very beginning and, therefore, the solutions in a pure rotational flow are the same as in a static fluid medium, namely

$$\sigma_{xx}(t) = \sigma_{yy}(t) = \frac{q}{2\pi}(4\pi q D t)^{1/q} \tag{22}$$

$$\sigma_{xy}(t) = 0 \tag{23}$$

or, in terms of the coordinates correlations,

$$\langle x^2 \rangle = \langle y^2 \rangle = \frac{1}{2\pi} \frac{q}{2q-1} (4\pi q D t)^{1/q} \tag{24}$$

$$\langle xy \rangle = 0 \tag{25}$$

so that we obtain $\langle \boldsymbol{x}^2 \rangle \sim t^{1/q}$ as in (3). Thus, the flow does not affect the characteristic exponent of diffusion.

2. Pure elongational flow ($\epsilon = 1$). For pure elongational flows we have the same differential equation and the same initial condition again both for σ_{xx} and for σ_{yy}, whence one necessarily has $\sigma_{xx}(t) = \sigma_{yy}(t)$ and the equations to be solved are

$$\frac{\mathrm{d}\sigma_{xx}}{\mathrm{d}t} = 2G\sigma_{xy} + 4\pi D \left(\frac{q}{2\pi}\right)^q (\sigma_{xx}^2 - \sigma_{xy}^2)^{\frac{1-q}{2}} \tag{26}$$

$$\frac{\mathrm{d}\sigma_{xy}}{\mathrm{d}t} = 2G\sigma_{xx}. \tag{27}$$

From (26) and (27) it is easy to verify that

$$(\sigma_{xx}^2 - \sigma_{xy}^2)^{\frac{q-1}{2}} d(\sigma_{xx}^2 - \sigma_{xy}^2) = 4\pi \frac{D}{G} \left(\frac{q}{2\pi}\right)^q d\sigma_{xy} \tag{28}$$

which is immediate to integrate and yields the functional relation between σ_{xx} and σ_{xy}. Using this relation in (26) one obtains the equation

$$\left(\frac{\mathrm{d}\sigma_{xy}}{\mathrm{d}t}\right)^2 = 4G^2\sigma_{xy}^2 + 4G^2 \left[2\pi(q+1)\frac{D}{G}\left(\frac{q}{2\pi}\right)^q \sigma_{xy}\right]^{\frac{2}{q+1}} \tag{29}$$

which is immediate after changing to $\boldsymbol{xi} = \left[2\pi(q+1)DG^{-1}\left(\frac{q}{2\pi}\right)^q\right]^{-\frac{1}{q+1}} \sigma_{xy}^{\frac{q}{1+q}}$. Imposing the initial conditions, we finally obtain the solutions

$$\sigma_{xx} = \sigma_{yy} = \frac{q}{4\pi}\left(\frac{4\pi Dq}{\omega}\right)^{1/q} \sinh(2\omega t)[\sinh(\omega t)]^{\frac{1-q}{q}} \tag{30}$$

$$\sigma_{xy} = \frac{q}{2\pi}\left(\frac{4\pi Dq}{\omega}\right)^{1/q} [\sinh(\omega t)]^{\frac{1+q}{q}} \tag{31}$$

where $\omega = \frac{2qG}{q+1}$.

For short times ($\omega t \ll 1$), the behaviour is

$$\sigma_{xx} = \frac{q}{2\pi}\left(\frac{4\pi Dq}{\omega}\right)^{1/q} (\omega t)^{1/q} \left(1 + \frac{3q+1}{6q}\omega^2 t^2 + \mathrm{O}(\omega^4 t^4)\right) \tag{32}$$

$$\sigma_{xy} = \frac{q}{2\pi}\left(\frac{4\pi Dq}{\omega}\right)^{1/q} (\omega t)^{\frac{1+q}{q}} \left(1 + \frac{1+q}{6q}\omega^2 t^2 + \mathrm{O}(\omega^4 t^4)\right) \tag{33}$$

whereas in the long-time limit ($\omega t \gg 1$) the dispersion grows exponentially as

$$\sigma_{xy} \simeq \sigma_{xx} \simeq \frac{q}{4\pi}\left(\frac{2\pi Dq}{\omega}\right)^{1/q} \exp\left(\frac{q+1}{q}\omega t\right). \tag{34}$$

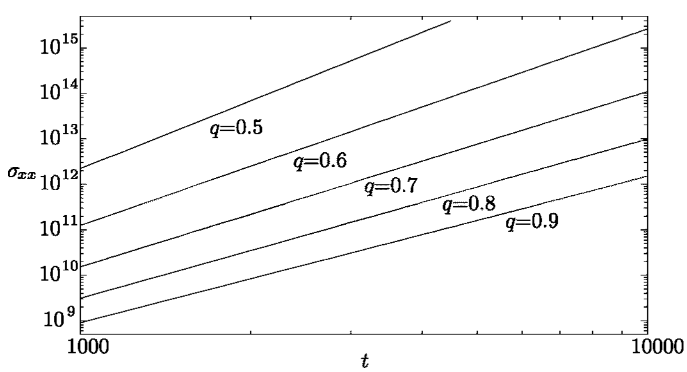

Figure 1. Log–log representation of the numerical results for σ_{xx} versus t. In the resolution we have taken $G = 1$ and $4\pi D\left(\frac{q}{2\pi}\right)^q = 1$.

Table 1. Comparison of the anomalous exponent for σ_{xx}, σ_{xy} and σ_{yy} in simple shear ($\epsilon = 0$) as computed with relations (36) and as found in a best fit analysis of the numerical results in two intervals of time: $100 < t < 1000$ and $1000 < t < 10\,000$. For the numerical resolution we take $G = 1$ and $4\pi D\left(\frac{q}{2\pi}\right)^q = 1$.

	σ_{xx}			σ_{xy}			σ_{yy}		
q	α_{th}	α_{100}	α_{1000}	β_{th}	β_{100}	β_{1000}	γ_{th}	γ_{100}	γ_{1000}
0.9	$\frac{29}{9}$	3.2176	3.2218	$\frac{20}{9}$	2.2191	2.2219	$\frac{11}{9}$	1.2205	1.2220
0.8	$\frac{7}{2}$	3.4907	3.4991	$\frac{5}{2}$	2.4934	2.4993	$\frac{3}{2}$	1.4960	1.4996
0.7	$\frac{27}{7}$	3.8420	3.8556	$\frac{20}{7}$	2.8459	2.8560	$\frac{13}{7}$	1.8498	1.8564
0.6	$\frac{13}{3}$	4.3098	4.3309	$\frac{10}{3}$	3.3153	3.3315	$\frac{7}{3}$	2.3207	2.3320
0.5	5	4.9633	4.9963	4	3.9707	3.9970	3	2.9779	2.9978
0.4	6	5.9392	5.9938	5	4.9494	4.9948	4	3.9594	3.9958
0.3	$\frac{23}{3}$	7.5528	7.6549	$\frac{20}{3}$	6.5679	6.6565	$\frac{17}{3}$	5.5825	5.6580
0.2	11	10.7285	10.9715	10	9.7537	9.9741	9	8.7778	8.9767
0.1	21	19.8456	20.8729	20	18.9024	19.8790	19	17.9553	18.8850

3. Simple shear ($\epsilon = 0$). For the case $\epsilon = 0$, the system of equations (19)–(21) does not admit a closed analytic solution and we have solved it numerically. Figure 1 is a log–log plot of σ_{xx} versus time for different values of q. The apparent linearity of the plot for long times seems to suggest that, for $t \gg G^{-1}$, the components of $\boldsymbol{\sigma}$ might follow a potential law. We then try the following solutions in (19)–(21)

$$\sigma_{xx} = at^{\alpha} \qquad \sigma_{xy} = bt^{\beta} \qquad \sigma_{yy} = ct^{\gamma} \tag{35}$$

and explore for what values of the parameters α, β and γ, the equations hold at sufficiently long times. Straightforward computations yield

$$\alpha = \frac{2+q}{q} \qquad \beta = \frac{2}{q} \qquad \gamma = \frac{2-q}{q} \tag{36}$$

which is in very reasonable agreement with the numerical results as might be seen from the data in table 1.

We therefore have that

$$\langle \boldsymbol{x}^2 \rangle \sim t^{\frac{2+q}{q}} \qquad \text{for } t \gg G^{-1} \tag{37}$$

for anomalous diffusion in a two-dimensional simple shear flow. Since $q < 1$, this represents an enhancement of diffusion with respect to standard diffusion in simple shear flows, where one has $\langle \boldsymbol{x}^2 \rangle \sim t^3$ at long times [10, 11].

If one now takes fractional derivatives instead of a nonlinear term to describe anomalous diffusion in a static fluid, any of the two following generalizations of the diffusion equation is possible

$$\frac{\partial^{\alpha} n}{\partial t^{\alpha}} = D\nabla^2 n \qquad 0 < \alpha < 1 \tag{38}$$

$$\frac{\partial n}{\partial t} = D\nabla^{2\mu} n \qquad 0 < \mu < 1 \tag{39}$$

where $\partial^{\alpha}/\partial t^{\alpha}$ stands for the Riemann–Liouville fractional derivative of order α and $\nabla^{2\mu}$ is minus the Riesz fractional derivative of order 2μ [8], which is defined as the inverse Fourier transform of $-k^{2\mu}$. In this paper we shall focus on the latter since the former brings about some mathematical difficulties, and it has recently been argued [12] that they are asymptotically equivalent, at least for bistable systems. The solutions of (39) for an initial delta distribution are Lévy distributions whose mean squared displacement is infinite but whose characteristic scaling is well defined [13, section 1.2.1.2], being of the form

$$\boldsymbol{x}^2 \sim t^{1/\mu}. \tag{40}$$

Turning now to diffusion in a simple shear flow, the fractional formalism leads one to the equation

$$\frac{\partial n}{\partial t} + Gy\frac{\partial n}{\partial x} = D\nabla^{2\mu} n \qquad 0 < \mu < 1. \tag{41}$$

Since the equations are linear and the field of velocity is wholly directed along the x-axis, we make the reasonable assumption that the scaling in the y-direction remains unchanged $y \sim t^{\frac{1}{2\mu}}$ so that we are led to the following equation in Fourier space

$$\frac{\partial \hat{n}}{\partial t} = [-\mathrm{i}at^{1/2\mu}k_x - D(k_x^2 + k_y^2)^{\mu}]\hat{n} \tag{42}$$

a being a constant. Equation (42) is easily integrable and we obtain the solution

$$n(\boldsymbol{x}, t) = \int \hat{n}(\boldsymbol{k}, t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{k} = \int \mathrm{e}^{-D|k|^{2\mu}t}\mathrm{e}^{\mathrm{i}(k_x x' + k_y y)}\,\mathrm{d}k_x\,\mathrm{d}k_y \tag{43}$$

with $x' = x - 2\mu a/(1+2\mu)t^{1+1/2\mu}$. Solution (43) is a stable distribution whose characteristic scaling is known to be (40), whence we obtain

$$x'^2 + y^2 \sim t^{1/\mu} \tag{44}$$

and making use of the definition of x' in terms of x and t it is straightforwardly concluded that

$$\boldsymbol{x}^2 \sim t^{\frac{1+2\mu}{\mu}}. \tag{45}$$

Now, it is interesting to compare this result with the one that we obtained in our analysis of nonlinear diffusion in a simple shear flow (37). Thus, in equilibrium, anomalous diffusion with $\langle x^2\rangle \sim t^{\nu}$ may be described either as the consequence of nonlinear diffusion (1) with $q = 1/\nu$, or of the formalism of fractional derivatives (39) with $\mu = 1/\nu$. However, the predictions concerning the change of the exponent of $\langle x^2\rangle$ in presence of simple shear are different. The nonlinear diffusion formalism studied here yields $\langle x^2\rangle \sim t^{2\nu+1}$ whereas the scheme based on fractional derivatives yields $\langle x^2\rangle \sim t^{\nu+2}$. Of course, for $\nu = 1$ (classical diffusion) both formalisms yield $\langle x^2\rangle \sim t^3$ in the presence of a simple shear flow. Thus, the experimental verification of the time behaviour of longitudinal displacements in a simple shear flow could be able, in principle, to discriminate which of both formalisms is more suitable as a description of anomalous diffusion in fluids. Of course, it is conceivable that one model could be more suitable for fluids and the other one for fractal solids (where some modifications should be made to include the two relevant parameters for diffusion in fractals [7]), so that these experiments would not completely eliminate either formalism.

It is also remarkable that, whereas the result for classical dispersion under simple shear flow $\langle x^2\rangle \sim t^3$ is analogous to Richardson's law for the dispersion of a tracer in a turbulent flow (for a review on turbulent diffusion refer to [9]), the nonlinear diffusion result $\langle x^2\rangle \sim t^{2\nu+1}$ with $\nu > 1$ bears a resemblance to the turbulent behaviour when one takes intermittency into account [14], in which $\langle x^2\rangle$ is no longer proportional to t^3 but to some fractional power slightly higher than 3 (for instance, in the β model, one would have $\langle x^2\rangle \sim t^{3.28}$ for an intermittency fractal dimension $D \simeq 2.83$). Whether this parallelism is just a coincidence or hints at a deeper underlying relation has not yet been explored.

As a conclusion, we have here completed the analysis of diffusion in linear shear flows by introducing anomalous diffusion in the form of a nonlinear diffusion term and by solving the equations for several incompressible stationary linear shear flows in two dimensions. The most interesting results have been found in a simple shear flow, which have enabled a preliminary comparison with the scheme of fractional derivatives. A more thorough analysis of anomalous diffusion in shear flows under the fractional picture is necessary in order to establish the detailed relation between these two different descriptions of anomalous diffusion and their convenience for diffusion in fluids.

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